Arithmetic and geometry of algebraic surfaces: the place of elliptic fibrations

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Plan for today

1. Arithmetic and algebraic geometry
2. Algebraic curves
3. Geometry of surfaces
4. Arithmetic of surfaces
5. Elliptic fibrations

Arithmetic and algebraic geometry

Algebraic Geometry: set of solutions of systems of polynomial equations (over an algebraically closed field, eg. $\mathbb{C}$) correspond to algebraic varieties.

Arithmetic/Diophantine Geometry: solutions over arbitrary fields or rings (eg. rationals, integers).

Notation:

$X(k)$ the solutions over k ($k-$points)

For simplicity in the rest of the talk: $k = \mathbb{Q}$

A prominent example

Fermat's last theorem (Taylor-Wiles, 1995):
If $(x, y, z) \in \mathbb{Z}^3$ is such that $x^n + y^n = z^n$, for some integer $n \geq 3$, then $xyz = 0$.

Geometry:
$C_n : x^n + y^n = z^n$ is a (plane, projective) algebraic curve of genus

$$g = \frac{(n-1)(n-2)}{2} \geq 1$$

# of holes in the associated Riemann surface $C_n(\mathbb{C})$.

A prominent example

Fermat's last theorem redux:

Let $n \geq 3$. The projective plane curve
$C_n : x^n + y^n = z^n$ has no $\mathbb{Q}$- points $(x : y : z) \in \mathbb{P}^2(\mathbb{Q})$,
with $x \cdot y \cdot z \neq 0$.

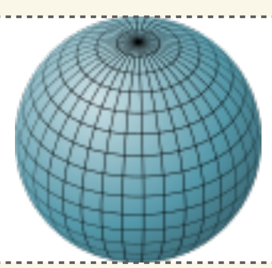
Geometry:

$C_n : x^n + y^n = z^n$ is a (plane, projective) algebraic curve of genus

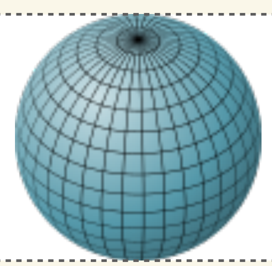
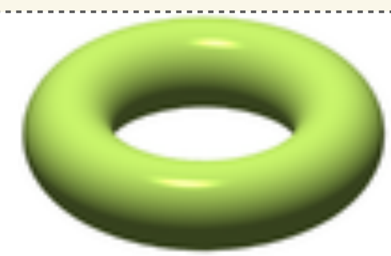
$$g = \frac{(n-1)(n-2)}{2} \geq 1$$

# of holes in the associated Riemann surface $C_n(\mathbb{C})$

Geometry determines arithmetic (of curves)

Genus/ rational points	$g = 0$	$g = 1$	$g > 1$
$X(\mathbb{Q}) =$ $(X(\mathbb{Q}) \neq \varnothing)$	$\mathbb{P}^1(k)$	**Has a GROUP structure**	finite
How/who?	Projection from a point	Euler (1750)	Faltings (1983)
Riemann surface $X(\mathbb{C})$			
Example	$x^2 + y^2 = z^2$	$x^3 + y^3 = z^3$	$x^4 + y^4 = z^4$

Geometry determines arithmetic (of curves)

Genus/ rational points	$g = 0$	$g = 1$	$g > 1$
$X(\mathbb{Q}) =$ $(X(\mathbb{Q}) \neq \varnothing)$	$\mathbb{P}^1(k)$	$T \oplus \mathbb{Z}^r$	finite
How/who?	Projection from a point	Mordell (1922)	Faltings (1983)
Riemann surface $X(\mathbb{C})$			
Example	$x^2 + y^2 = z^2$	$x^3 + y^3 = z^3$	$x^4 + y^4 = z^4$

What about surfaces?

For example, the set of solutions of

$$x^n + y^n = z^n + w^n.$$

Surfaces

Some guiding questions

Surfaces

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$$X(k)$$

Surfaces

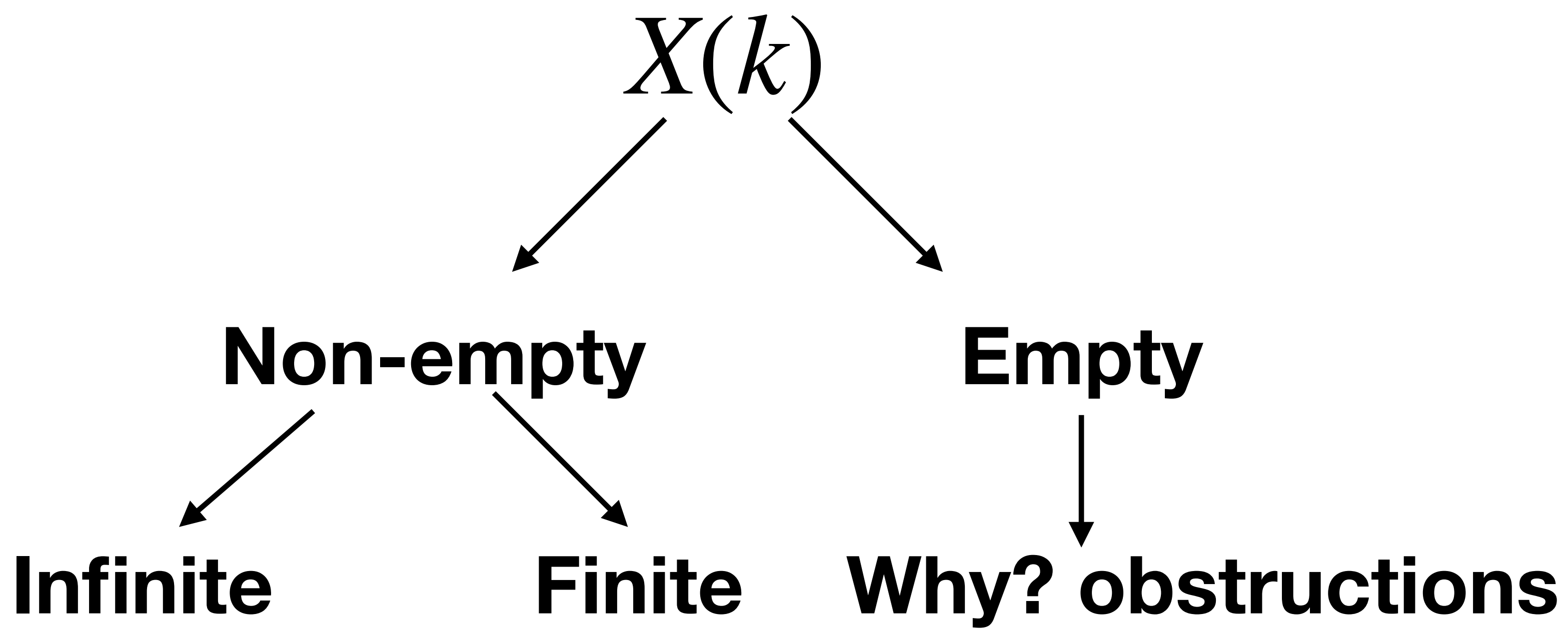
Some guiding questions

$$X(k)$$

Non-empty **Empty**

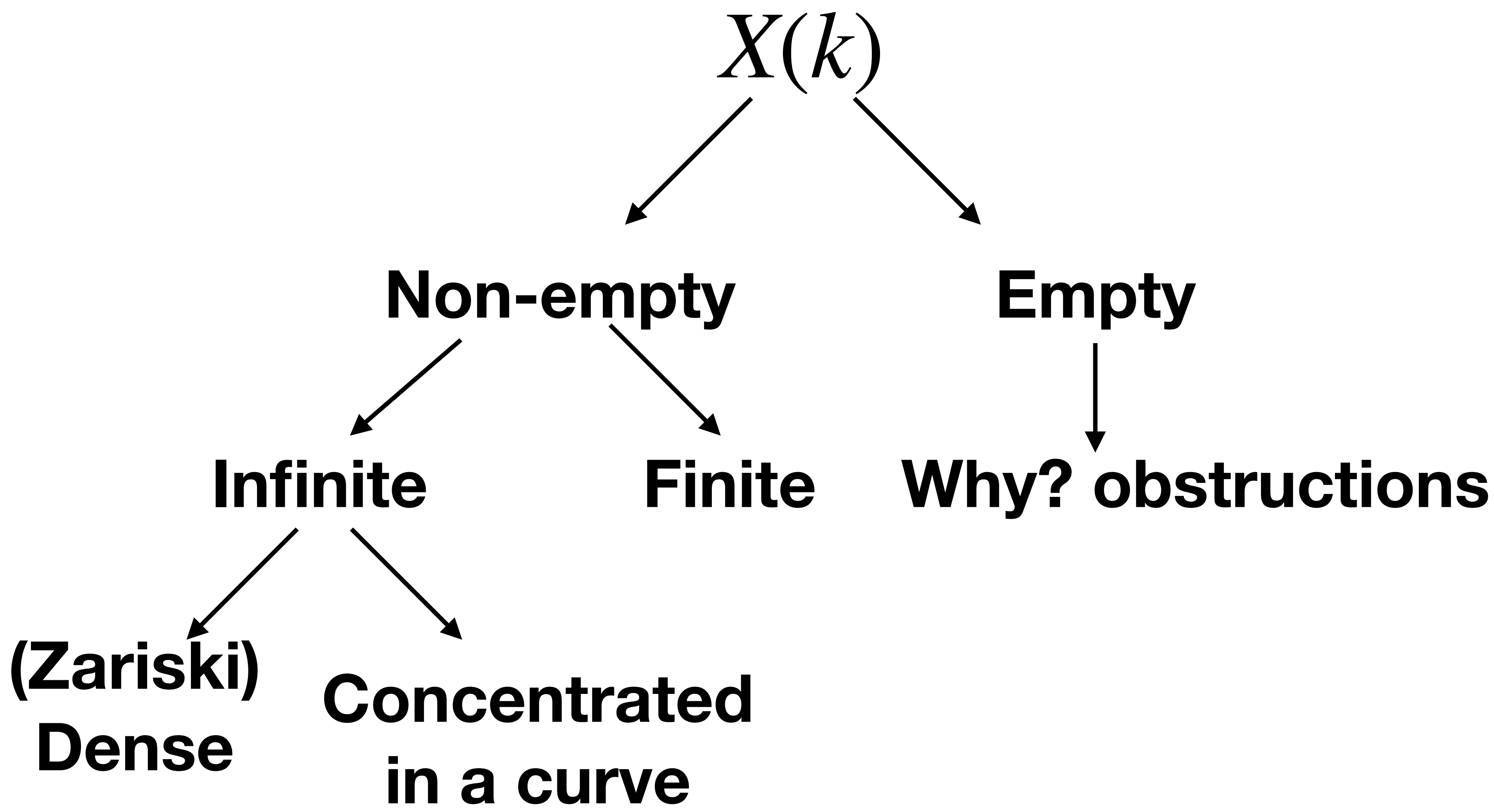
Surfaces

Some guiding questions



Surfaces

Some guiding questions



Surfaces

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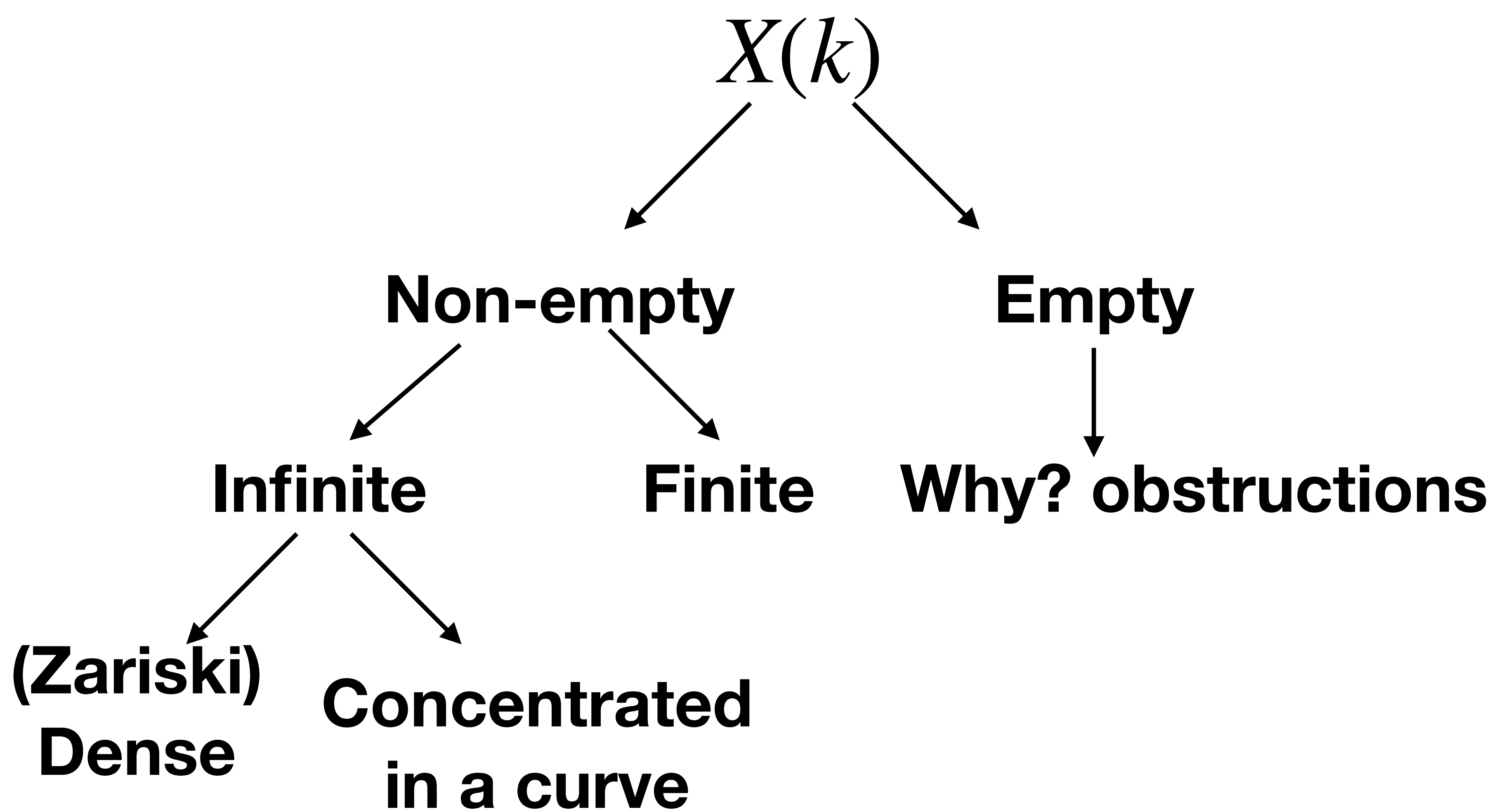
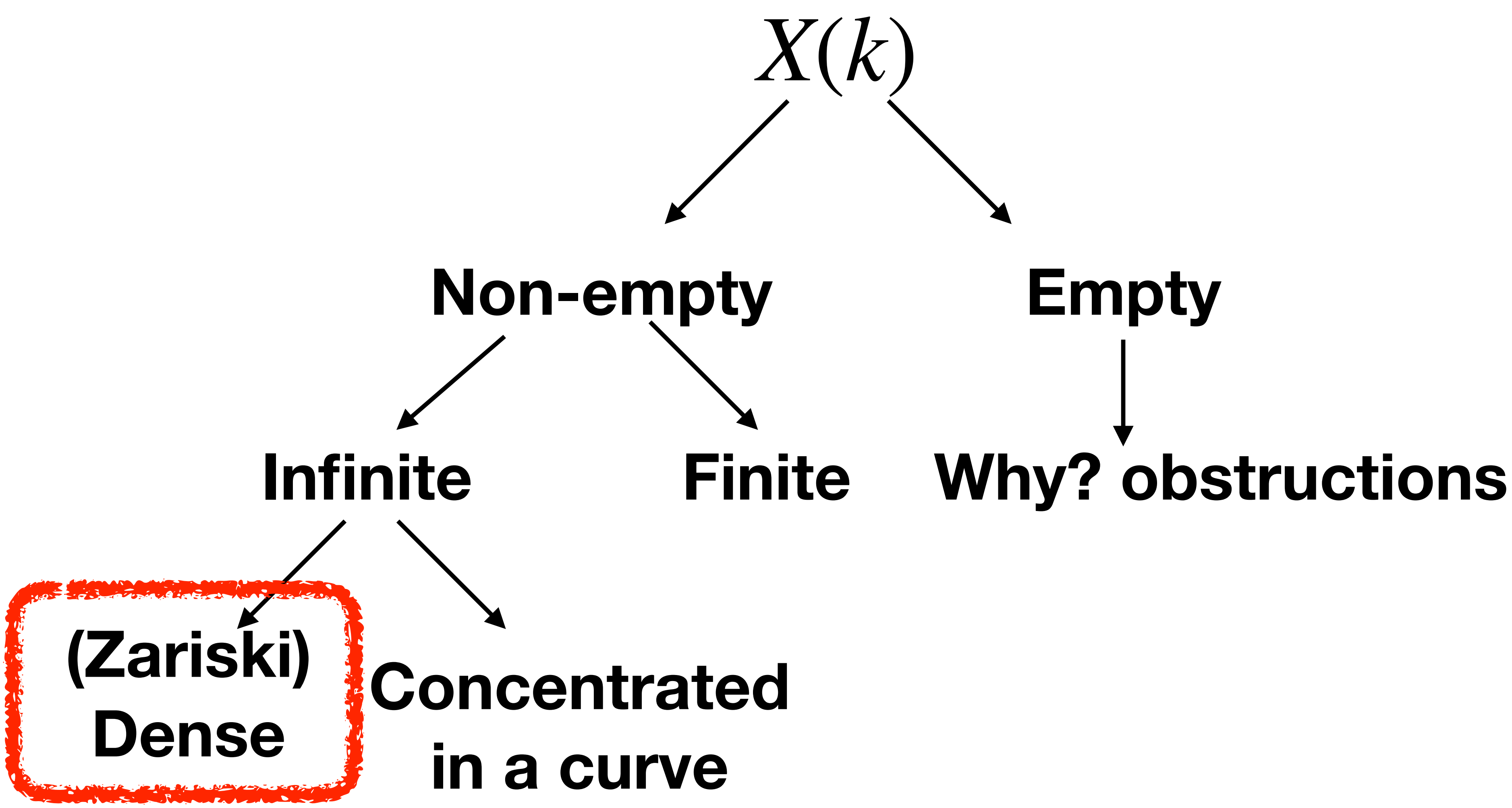


Does geometry determine their arithmetic?

Surfaces

Some guiding questions



Does geometry determine their arithmetic?

The Kodaira dimension

Let K_X be the canonical divisor of X.

If $h^0(X, nK_X)$ does not vanish for all positive integers n, then there is a unique integer $\kappa = \kappa(X)$ with $0 \leq \kappa \leq d$ such that:

$$\limsup_{n\to\infty} \frac{h^0(X, nK_X)}{n^\kappa}$$ exists and is non-zero.

Definition: The integer $\kappa(X)$ is called the *Kodaira dimension* of X.

We set $\kappa(X) = -\infty$ if $h^0(X, nK_X)$ vanishes for all n.

The Kodaira dimension of a curve

$\kappa(C) = -\infty$, **if** $g(C) = 0$

$\kappa(C) = 0$, **if** $g(C) = 1$

$\kappa(C) = 1$, **if** $g(C) > 1$

Geometry determines arithmetic (of curves)

Kodaira dim/ rational points	$\kappa = -\infty$	$\kappa = 0$	$\kappa = 1$
$X(\mathbb{Q}) =$ $(X(\mathbb{Q}) \neq \varnothing)$	$\mathbb{P}^1(k)$	$T \oplus \mathbb{Z}^r$	finite
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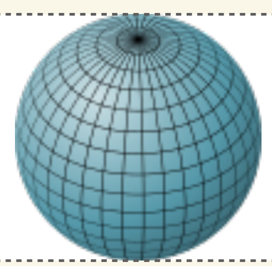
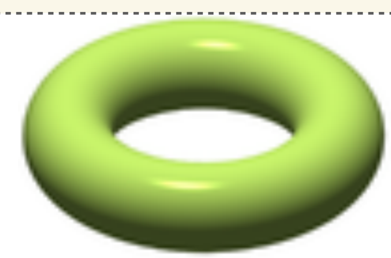
Riemann surface $X(\mathbb{C})$			
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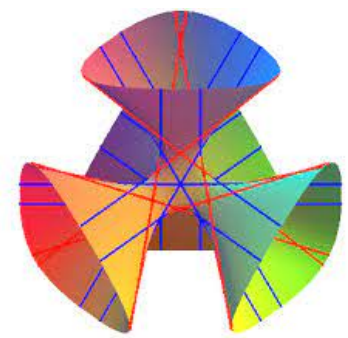
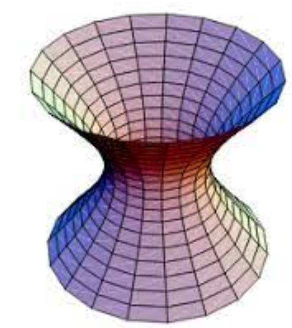
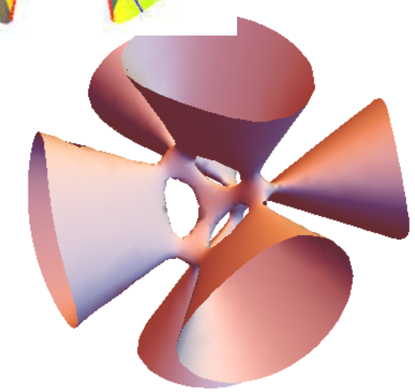
Classification of algebraic surfaces (Enriques and Kodaira): divide an conquer

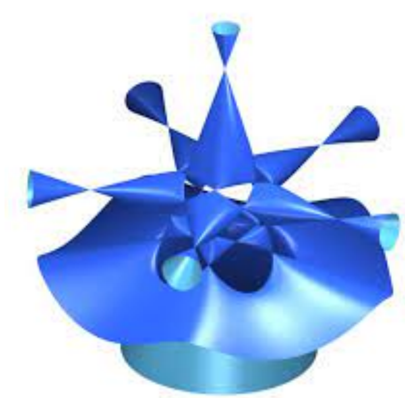
Kodaira dimension	Surfaces
$-\infty$	Rational or $C \times \mathbb{P}^1$
0	K3, Enriques, Bielliptic, Abelian
1	Honest elliptic
2	General type



or







Classification of algebraic surfaces (Enriques and Kodaira): divide an conquer

Kodaira dimension	Surfaces	A Fermat type equation	There is k for which X(k) dense	X(k) not dense for any k
$-\infty$	Rational or $C \times \mathbb{P}^1$	$x^2 + y^2 + z^2 = w^2$ **or** $x^3 + y^3 + z^3 = w^3$	**many examples**	$C \times \mathbb{P}^1, g(C) > 1$
0	K3, Enriques, Bielliptic, Abelian	$x^4 + y^4 = z^4 + w^4$	**many examples**	**no example**
1	Honest elliptic		**many examples**	$E \times C$ $g(E) = 1, g(C) > 1$
2	General type	$x^n + y^n = z^n + w^n$ $n \geq 5$	**no example**	**many examples**

Table by Lucia Caporaso

Density of $k-$points: state of the art

- $\kappa = -\infty$: Rational surfaces admit $k-$minimal models that are either del Pezzo surfaces or conic bundles. A lot is known (S. -Testa- Várilly-Alvarado, S. - van Luijk) but there are still open cases!

- $\kappa = 0$: **Campana's conjecture:** $k-$points are potentially dense. Known for many surfaces.

- $\kappa = 1$: Admit a genus one fibration. If there is a non-torsion section then $k-$points form a dense set.

- $\kappa = 2$: **Bombieri-Lang conjecture**: $k'-$points are not dense for any k'/k finite extension.

Density of $k-$points: Techniques

Goal: generate new points from existing ones.

How?

Density of $k-$points: Techniques

Goal: generate new points from existing ones.

How?

- Apply automorphisms defined over the ground field (e.g. arising from the group law on an elliptic curve);

OR/AND

- Look for subvarieties that are expected to have many rational points.

Elliptic fibrations

curves to understand surfaces

Given a surface X, an elliptic fibration on X is a surjective morphism to a curve, $\pi : X \to B$, such that:

- almost all fibers are smooth curves of genus 1
- there are singular fibers
- there is a section ($\Rightarrow$ fibres are elliptic curves!)

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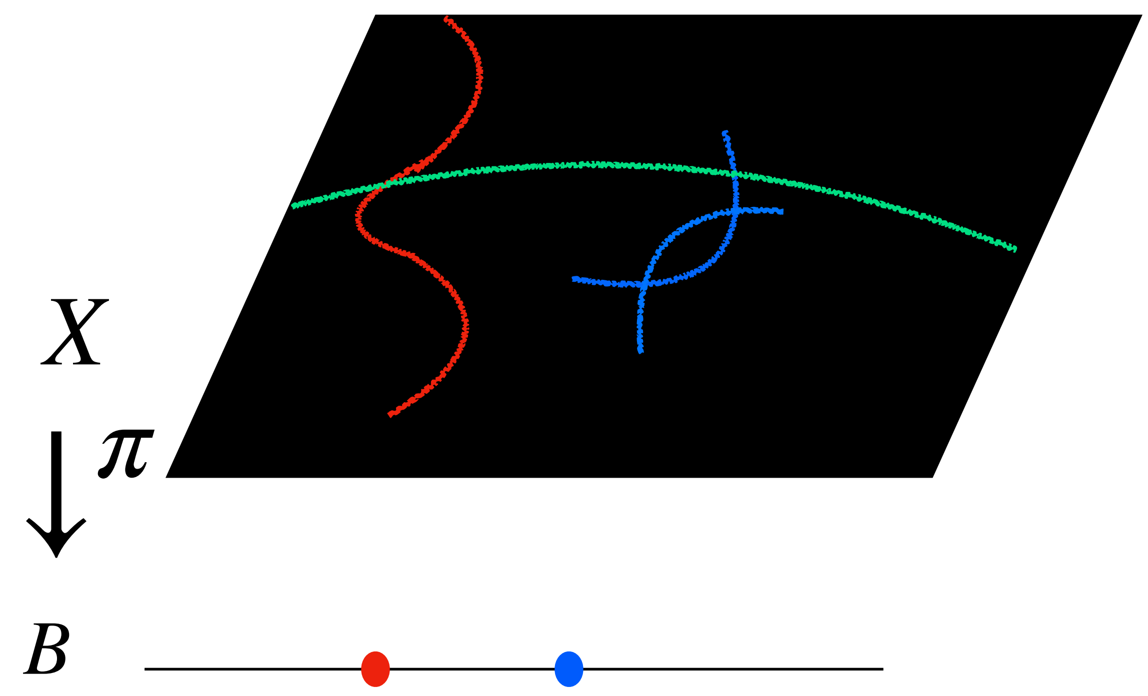

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Examples

a) A family of plane cubics: $y^2 = x^3 + tx + t$.

$$X \to \mathbb{P}^1_t$$

b) The surface described by the equations:

$$x^4 + y^4 = z^4 + w^4.$$

Examples

a) A family of plane cubics: $y^2 = x^3 + tx + t$.

$$X \to \mathbb{P}^1_t$$

b) The surface described by the equations:

$$z^2 + y^2 = t(x^2 - w^2)$$

$$t(z^2 - y^2) = x^2 + w^2.$$

Why do we care?

A. Density of rational points (**S**.- van Luijk, Bogomolov-Tschinkel)

B. Unirationality of conic bundles (Kóllar-Mella)

C. Useful to find elliptic curves with high rank (Elkies)

D. Shioda-Tate formula (helps understand geometry)

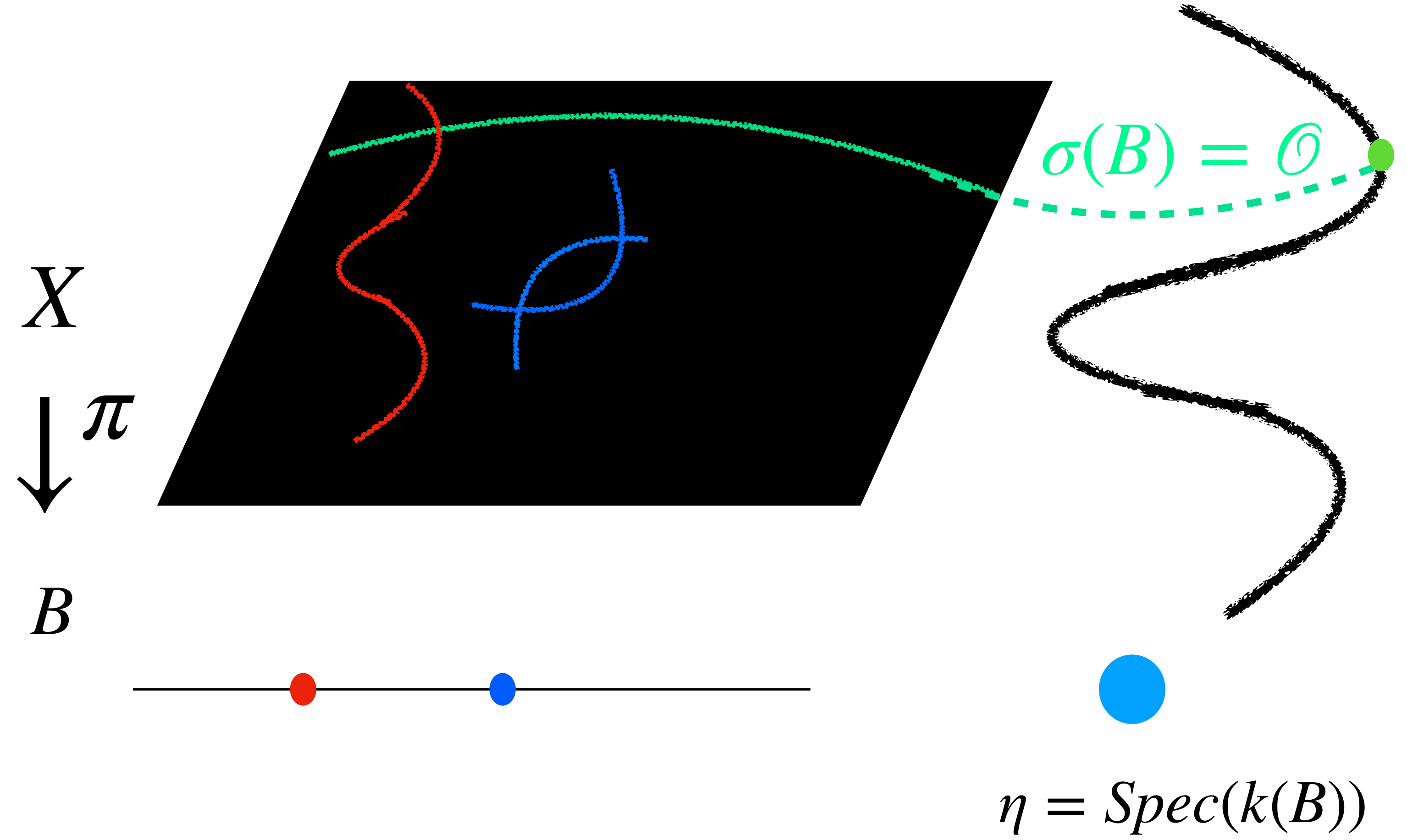
E. Sphere packing (Elkies, Shioda)

F. Error-correcting codes (**S**. - Várilly-Alvarado - Voloch)

And more….

Elliptic fibrations and rational points

Let k be a number field and $\pi : X \to \mathbb{P}^1$ an elliptic surface over k.



Elliptic fibrations and rational points

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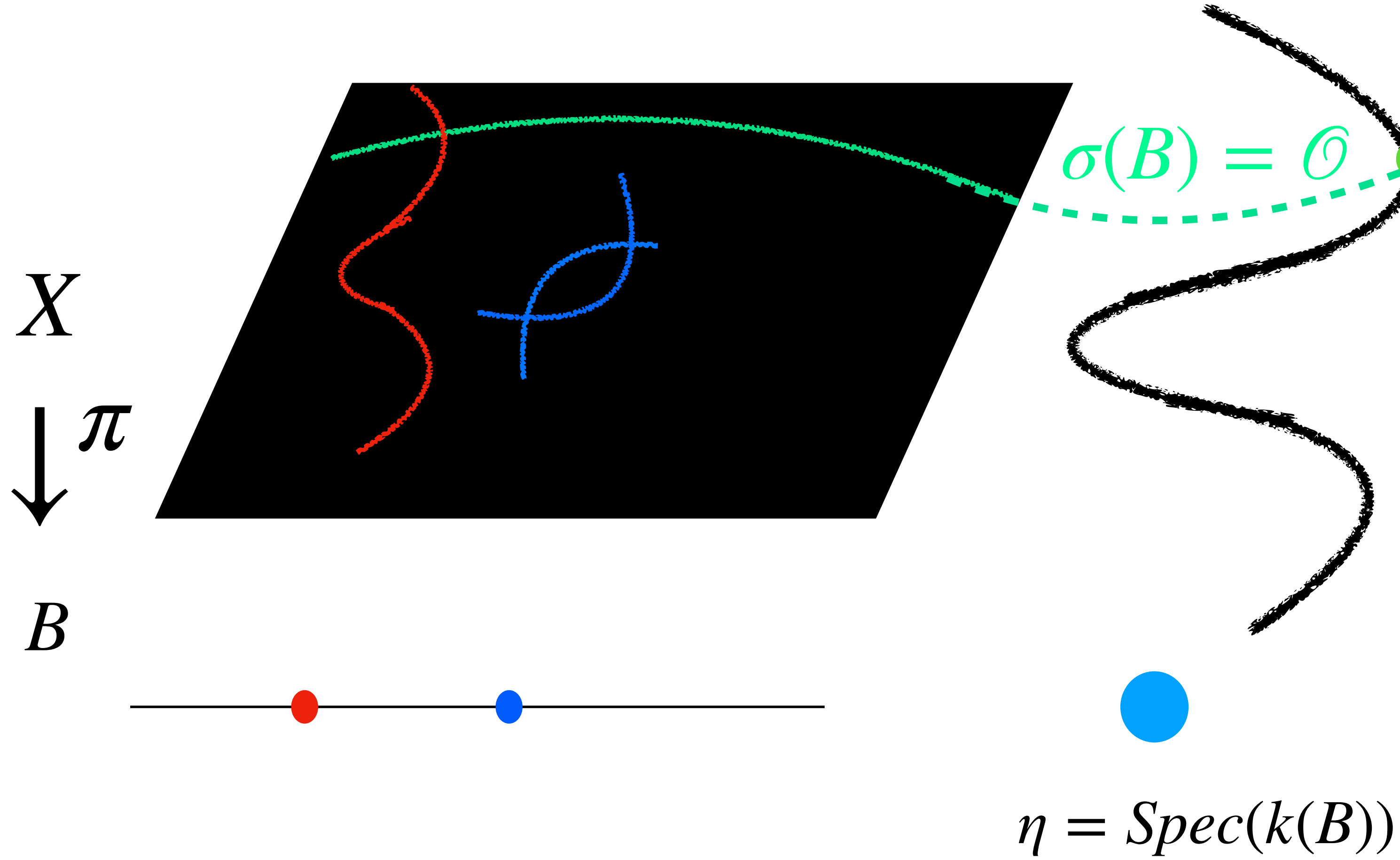
Mordell-Weil: For $t \in B(k)$,
$\pi^{-1}(t)(k) \cong \mathbb{Z}^{r_t} \oplus \mathrm{Tors}_t$

Lang-Néron: For η,
$\pi^{-1}(\eta)(k(B)) \cong \mathbb{Z}^{r} \oplus \mathrm{Tors}$

Silverman's specialization:
$r_t \geq r$ for all but finitely many $t \in B(k)$.



Conclusion

If $r > 0$ then $X(k)$ is Zariski dense in X.

Elliptic fibrations and rational points

Let k be a number field and $\pi : X \to \mathbb{P}^1$ an elliptic surface over k.
Let $\mathcal{F}(k) := \{t \in \mathbb{P}^1(k); r_t > 0\}$.

Theorem: $X(k)$ is Zariski dense in $X \iff \#\mathcal{F} = \infty$.

Sketch of the proof:

($\Rightarrow$) Mérel's result on the uniform boundedness of torsion.

($\Leftarrow$) If $X(k)$ contained in a finite union of curves then, in particular, it is contained in a finite union of multisections and fibers. Each multisection intersects a given fiber in a finite number of points. Hence all but finitely many fibers would have rank 0.

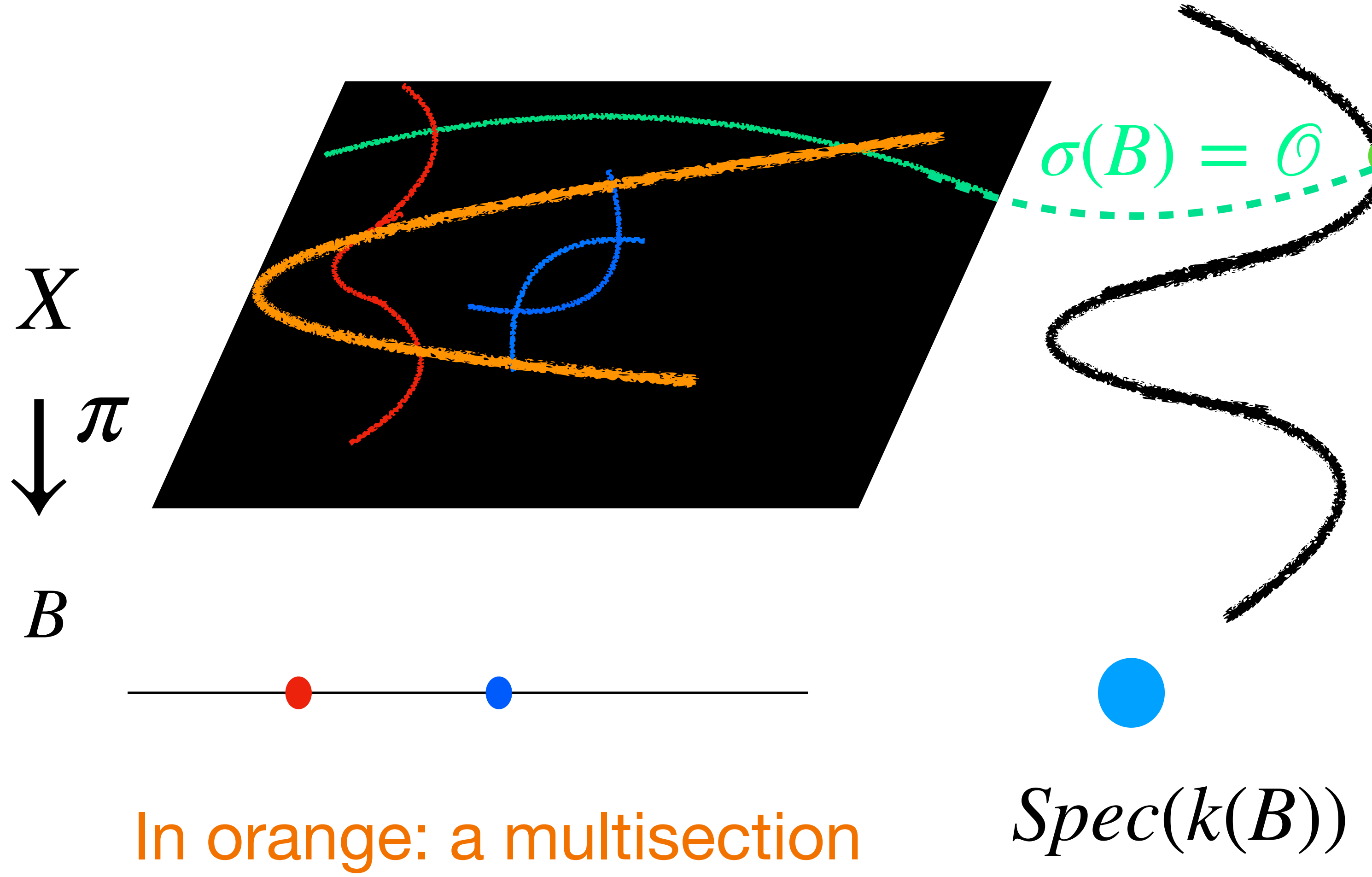
How do we show that $\mathscr{F}(k)$ is infinite?

Elliptic fibrations and rational points

Let k be a number field and $\pi : X \to \mathbb{P}^1$ an elliptic surface over k.

Method A (e.g. Bogomolov-Tschinkel, S.-vanLuijk): Find a non-torsion multisection C/k s.t. $\#C(k) = \infty$.

Method B (Rohrlich): Variation of root number. Subject to BSD and Parity conjecture



Thank you!
Danke!